Mathematics

Mathematics (colloquially, **maths** or **math**), is the body of knowledge centered on such concepts as quantity, structure, space, and change, and also the academic discipline that studies them. Benjamin Peirce called it "the science that draws necessary conclusions". Other practitioners of mathematics maintain that mathematics is the science of pattern, that mathematicians seek out patterns whether found in numbers, space, science, computers, imaginary abstractions, or elsewhere. Mathematicians explore such concepts, aiming to formulate new conjectures and establish their truth by rigorous deduction from appropriately chosen axioms and definitions.

Through the use of abstraction and logical reasoning, mathematics evolved from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Knowledge and use of basic mathematics have always been an inherent and integral part of individual and group life. Refinements of the basic ideas are visible in mathematical texts originating in ancient Egypt, Mesopotamia, ancient India, ancient China, and ancient Greece. Rigorous arguments first appear in Euclid's *Elements*. The development continued in fitful bursts until the Renaissance period of the 16th century, when mathematical innovations interacted with new scientific discoveries, leading to an acceleration in research that continues to the present day.

Today, mathematics is used throughout the world in many fields, including natural science, engineering, medicine, and the social sciences such as economics. Applied mathematics, the application of mathematics to such fields, inspires and makes use of new mathematical discoveries and sometimes leads to the development of entirely new disciplines. Mathematicians also engage in pure mathematics, or mathematics for its own sake, without having any application in mind, although applications for what began as pure mathematics are often discovered later.

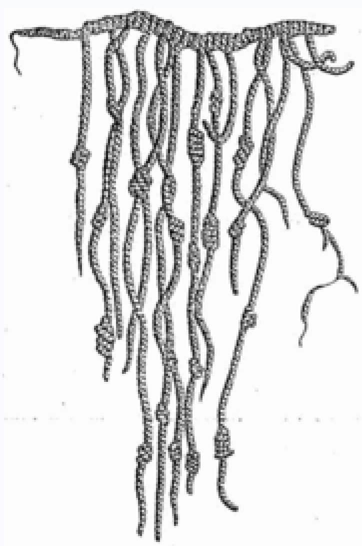
Etymology

The word "mathematics" (Greek: μαθηματικά or *mathēmatiká*) comes from the Greek μάθημα (*máthēma*), which means *learning*, *study*, *science*, and additionally came to have the narrower and more technical meaning "mathematical study", even in Classical times. Its adjective is μαθηματικός (*mathēmatikós*), *related to learning*, or *studious*, which likewise further came to mean *mathematical*. In particular, μαθηματικὴ τέχνη (*mathēmatikḗ tékhnē*), in Latin *ars mathematica*, meant *the mathematical art*.

The apparent plural form in English, like the French plural form *les mathématiques* (and the less commonly used singular derivative *la mathématique*), goes back to the Latin neuter plural *mathematica* (Cicero), based on the Greek plural τα (*ta mathēmatiká*), used by Aristotle, and meaning roughly "all things mathematical". In

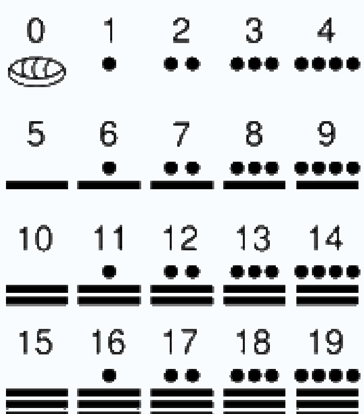
English, however, *mathematics* is a singular noun, often shortened to *math* in English-speaking North America and *maths* elsewhere.

History



The evolution of mathematics might be seen as an ever-increasing series of abstractions, or alternatively an expansion of subject matter. The first abstraction was probably that of numbers. The realization that two apples and two oranges have something in common was a breakthrough in human thought. In addition to recognizing how to count *physical* objects, prehistoric peoples also recognized how to count *abstract* quantities, like time — days, seasons, years. Arithmetic (addition, subtraction, multiplication and division), naturally followed. Monolithic monuments testify to knowledge of geometry.

Further steps need writing or some other system for recording numbers such as tallies or the knotted strings called quipu used by the Inca empire to store numerical data. Numeral systems have been many and diverse.



Mayan numerals

From the beginnings of recorded history, the major disciplines within mathematics arose out of the need to do calculations relating to taxation and commerce, to understand the relationships among numbers, to measure land, and to predict astronomical events. These needs can be roughly related to the broad subdivision of mathematics, into the studies of *quantity*, *structure*, *space*, and *change*.

Mathematics has since been greatly extended, and there has been a fruitful interaction between mathematics and science, to the benefit of both. Mathematical discoveries have been made throughout history and continue to be made today. According to Mikhail B. Sevryuk, in the January 2006 issue of the Bulletin of the American Mathematical Society, "The number of papers and books included in the Mathematical Reviews database since 1940 (the first year of operation of MR) is now more than 1.9 million, and more than 75 thousand items are added to the database each year. The overwhelming majority of works in this ocean contain new mathematical theorems and their proofs."

Inspiration, pure and applied mathematics, and aesthetics

Sir Isaac Newton (1643-1727), an inventor of infinitesimal calculus.

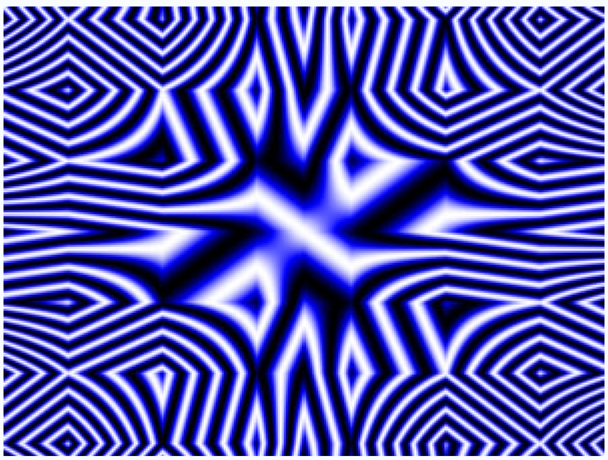
Main article: Mathematical beauty

Mathematics arises wherever there are difficult problems that involve quantity, structure, space, or change. At first these were found in commerce, land measurement and later astronomy; nowadays, all sciences suggest problems studied by mathematicians, and many problems arise within mathematics itself. Newton was one of the infinitesimal calculus inventors, Feynman invented the Feynman path integral using a combination of reasoning and physical insight, and today's string theory also inspires new mathematics. Some mathematics is only relevant in the area that inspired it, and is applied to solve further problems in that area. But often mathematics inspired by one area proves useful in many areas, and joins the general stock of mathematical concepts. The remarkable fact that even the "purest" mathematics often turns out to have practical applications is what Eugene Wigner has called "the unreasonable effectiveness of mathematics."

As in most areas of study, the explosion of knowledge in the scientific age has led to specialization in mathematics. One major distinction is between pure mathematics and applied mathematics. Several areas of applied mathematics have merged with related traditions outside of mathematics and become disciplines in their own right, including statistics, operations research, and computer science.

For those who are mathematically inclined, there is often a definite aesthetic aspect to much of mathematics. Many mathematicians talk about the *elegance* of mathematics, its intrinsic aesthetics and inner beauty. Simplicity and generality are valued. There is beauty in a simple and elegant proof, such as Euclid's proof that there are infinitely many prime numbers, and in an elegant numerical method that speeds calculation, such as the fast Fourier transform. G. H. Hardy in *A Mathematician's Apology* expressed the belief that these aesthetic considerations are, in themselves, sufficient to justify the study of pure mathematics. Mathematicans often strive to find proofs of theorems that are particularly elegant, a quest Paul Erdős often referred to as finding proofs from "The Book" in which God had written down his favorite proofs. The popularity of recreational mathematics is another sign of the pleasure many find in solving mathematical questions.

Notation, language, and rigor



In modern notation, simple expressions can describe complex concepts. This image is the graph of $\cos(y \arccos \sin|x| + x \arcsin \cos|y|)$.

Most of the mathematical notation in use today was not invented until the 16th century. Before that, mathematics was written out in words, a painstaking process that limited mathematical discovery. Modern notation makes mathematics much easier for the professional, but beginners often find it daunting. It is extremely compressed: a few symbols contain a great deal of information. Like musical notation, modern mathematical notation has a strict syntax and encodes information that would be difficult to write in any other way.

Mathematical language also is hard for beginners. Words such as *or* and *only* have more precise meanings than in everyday speech. Also confusing to beginners, words such as *open* and *field* have been given specialized mathematical meanings. Mathematical jargon includes technical terms such as *homeomorphism* and *integrable*. But there is a reason for special notation and technical jargon: mathematics requires more precision than everyday speech. Mathematicians refer to this precision of language and logic as "rigor".

Rigor is fundamentally a matter of mathematical proof. Mathematicians want their theorems to follow from axioms by means of systematic reasoning. This is to avoid mistaken "theorems", based on fallible intuitions, of which many instances have occurred in the history of the subject. The level of rigor expected in mathematics has varied over time: the Greeks expected detailed arguments, but at the time of Isaac Newton the methods employed were less rigorous. Problems inherent in the definitions used by Newton would lead to a resurgence of careful analysis and formal proof in the 19th century. Today, mathematicians continue to argue among themselves about computer-assisted proofs. Since large computations are hard to verify, such proofs may not be sufficiently rigorous. Axioms in traditional thought were "self-evident truths", but that conception is problematic. At a formal level, an axiom is just a string of symbols, which has an intrinsic meaning only in the context of all derivable formulas of an axiomatic system. It was the goal of Hilbert's program to put all of mathematics on a firm axiomatic basis, but according to Gödel's incompleteness theorem every (sufficiently powerful) axiomatic system has undecidable formulas; and so a final axiomatization of mathematics is impossible. Nonetheless mathematics is often imagined to be (as far as its formal content) nothing but set theory in some axiomatization, in the sense that every mathematical statement or proof could be cast into formulas within set theory.

Mathematics as science

Carl Friedrich Gauss, himself known as the "prince of mathematicians", referred to mathematics as "the Queen of the Sciences".

Carl Friedrich Gauss referred to mathematics as "the Queen of the Sciences". In the original Latin *Regina Scientiarum*, as well as in German *Königin der Wissenschaften*, the word corresponding to *science* means (field of) knowledge. Indeed, this is also the original meaning in English, and there is no doubt that mathematics is in this sense a science. The specialization restricting the meaning to *natural* science is of later date. If one considers science to be strictly about the physical world, then mathematics, or at least pure mathematics, is not a science. Albert Einstein has stated that *"as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."*

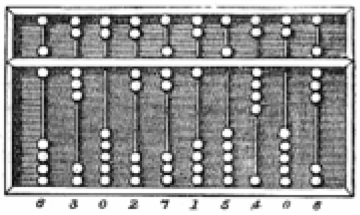
Many philosophers believe that mathematics is not experimentally falsifiable,
and thus not a science according to the definition of Karl Popper. However, in the 1930s important work in mathematical logic showed that mathematics cannot be reduced to logic, and Karl Popper concluded that "most mathematical theories are, like those of physics and biology, hypothetico-deductive: pure mathematics therefore turns out to be much closer to the natural sciences whose hypotheses are conjectures, than it seemed even recently." Other thinkers, notably Imre Lakatos, have applied a version of falsificationism to mathematics itself.

An alternative view is that certain scientific fields (such as theoretical physics) are mathematics with axioms that are intended to correspond to reality. In fact, the theoretical physicist, J. M. Ziman, proposed that science is *public knowledge* and thus includes mathematics. In any case, mathematics shares much in common with many fields in the physical sciences, notably the exploration of the logical consequences of assumptions. Intuition and experimentation also play a role in the formulation of conjectures in both mathematics and the (other) sciences. Experimental mathematics continues to grow in importance within mathematics, and computation and simulation are playing an increasing role in both the sciences and mathematics, weakening the objection that mathematics does not use the scientific method. In his 2002 book *A New Kind of Science*, Stephen Wolfram argues that computational mathematics deserves to be explored empirically as a scientific field in its own right.

The opinions of mathematicians on this matter are varied. While some in applied mathematics feel that they are scientists, those in pure mathematics often feel that they are working in an area more akin to logic and that they are, hence, fundamentally philosophers. Many mathematicians feel that to call their area a science is to downplay the importance of its aesthetic side, and its history in the traditional seven liberal arts; others feel that to ignore its connection to the sciences is to turn a blind eye to the fact that the interface between mathematics and its applications in science and engineering has driven much development in mathematics. One way this difference of viewpoint plays out is in the philosophical debate as to whether mathematics is *created* (as in art) or *discovered* (as in science). It is common to see universities divided into sections that include a division of *Science and Mathematics*, indicating that the fields are seen as being allied but that they do not coincide. In practice, mathematicians are typically grouped with scientists at the gross level but separated at finer levels. This is one of many issues considered in the philosophy of mathematics.

Mathematical awards are generally kept separate from their equivalents in science. The most prestigious award in mathematics is the Fields Medal, established in 1936 and now awarded every 4 years. It is often considered, misleadingly, the equivalent of science's Nobel Prizes. The Wolf Prize in Mathematics, instituted in 1979, recognizes lifetime achievement, and another major international award, the Abel Prize, was introduced in 2003. These are awarded for a particular body of work, which may be innovation, or resolution of an outstanding problem in an established field. A famous list of 23 such open problems, called "Hilbert's problems", was compiled in 1900 by German mathematician David Hilbert. This list achieved great celebrity among mathematicians, and at least nine of the problems have now been solved. A new list of seven important problems, titled the "Millennium Prize Problems", was published in 2000. Solution of each of these problems carries a $1 million reward, and only one (the Riemann hypothesis) is duplicated in Hilbert's problems.

Fields of mathematics



An abacus, a simple calculating tool used since ancient times

As noted above, the major disciplines within mathematics first arose out of the need to do calculations in commerce, to understand the relationships between numbers, to measure land, and to predict astronomical events. These four needs can be roughly related to the broad subdivision of mathematics into the study of quantity, structure, space, and change (i.e., arithmetic, algebra, geometry, and analysis). In addition to these main concerns, there are also subdivisions dedicated to exploring links from the heart of mathematics to other fields: to logic, to set theory (foundations), to the empirical mathematics of the various sciences (applied mathematics), and more recently to the rigorous study of uncertainty.

Quantity

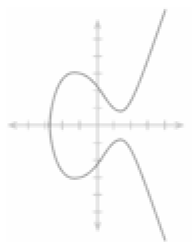
The study of quantity starts with numbers, first the familiar natural numbers and integers ("whole numbers") and arithmetical operations on them, which are characterized in arithmetic. The deeper properties of integers are studied in number theory, whence such popular results as Fermat's last theorem. Number theory also holds two widely-considered unsolved problems: the twin prime conjecture and Goldbach's conjecture.

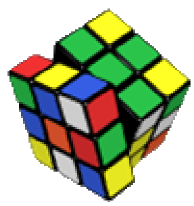
As the number system is further developed, the integers are recognized as a subset of the rational numbers ("fractions"). These, in turn, are contained within the real numbers, which are used to represent continuous quantities. Real numbers are generalized to complex numbers. These are the first steps of a hierarchy of numbers that goes on to include quarternions and octonions. Consideration of the natural numbers also leads to the transfinite numbers, which formalize the concept of counting to infinity. Another area of study is size, which leads to the cardinal numbers and then to another conception of infinity: the aleph numbers, which allow meaningful comparison of the size of infinitely large sets.

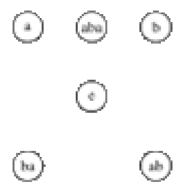
$1, 2, 3$	$-2, -1, 0, 1, 2$	$-2, \frac{2}{3}, 1.21$	$-e, \sqrt{2}, 3, \pi$	$2, i, -2+3i, 2e^{i\frac{4\pi}{3}}$
Natural numbers	Integers	Rational numbers	Real numbers	Complex numbers

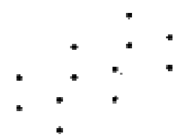
Structure

Many mathematical objects, such as sets of numbers and functions, exhibit internal structure. The structural properties of these objects are investigated in the study of groups, rings, fields and other abstract systems, which are themselves such objects. This is the field of abstract algebra. An important concept here is that of vectors, generalized to vector spaces, and studied in linear algebra. The study of vectors combines three of the fundamental areas of mathematics: quantity, structure, and space. Vector calculus expands the field into a fourth fundamental area, that of change.


Number theory

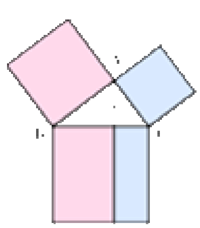

Abstract algebra

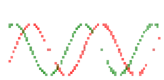

Group theory

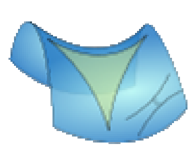

Order theory

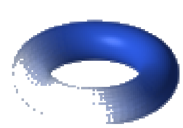
Space

The study of space originates with geometry - in particular, Euclidean geometry. Trigonometry combines space and numbers, and encompasses the well-known Pythagorean theorem. The modern study of space generalizes these ideas to include higher-dimensional geometry, non-Euclidean geometries (which play a central role in general relativity) and topology. Quantity and space both play a role in analytic geometry, differential geometry, and algebraic geometry. Within differential geometry are the concepts of fiber bundles and calculus on manifolds. Within algebraic geometry is the description of geometric objects as solution sets of polynomial equations, combining the concepts of quantity and space, and also the study of topological groups, which combine structure and space. Lie groups are used to study space, structure, and change. Topology in all its many ramifications may have been the greatest growth area in 20th century mathematics, and includes the long-standing Poincaré conjecture and the controversial four color theorem, whose only proof, by computer, has never been verified by a human.





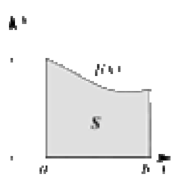
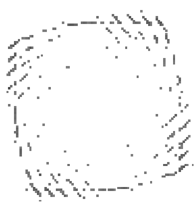
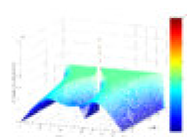
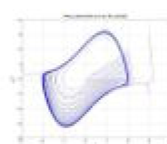
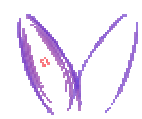




Geometry	Trigonometry	Differential geometry	Topology	Fractal geometry

Change

Understanding and describing change is a common theme in the natural sciences, and calculus was developed as a powerful tool to investigate it. Functions arise here, as a central concept describing a changing quantity. The rigorous study of real numbers and real-valued functions is known as real analysis, with complex analysis the equivalent field for the complex numbers. The Riemann hypothesis, one of the most fundamental open questions in mathematics, is drawn from complex analysis. Functional analysis focuses attention on (typically infinite-dimensional) spaces of functions. One of many applications of functional analysis is quantum mechanics. Many problems lead naturally to relationships between a quantity and its rate of change, and these are studied as differential equations. Many phenomena in nature can be described by dynamical systems; chaos theory makes precise the ways in which many of these systems exhibit unpredictable yet still deterministic behavior.

				
Calculus	Vector calculus	Differential equations	Dynamical systems	Chaos theory

Foundations and philosophy

In order to clarify the foundations of mathematics, the fields of mathematical logic and set theory were developed, as well as category theory which is still in development.

Mathematical logic is concerned with setting mathematics on a rigid axiomatic framework, and studying the results of such a framework. As such, it is home to Gödel's second incompleteness theorem, perhaps the most widely celebrated result in logic, which (informally) implies that any formal system that contains basic arithmetic, if *sound* (meaning that all theorems that can be proven are true), is necessarily *incomplete* (meaning that there are true theorems which cannot be proved *in that system*). Gödel showed how to construct, whatever the given collection of number-theoretical axioms, a formal statement in the logic that is a true number-theoretical fact, but which does not follow from those axioms. Therefore no formal system is a true axiomatization of full number theory. Modern logic is divided into recursion theory, model theory, and proof theory, and is closely linked to theoretical computer science.

Mathematical logic	Set theory	Category theory

Discrete mathematics

Discrete mathematics is the common name for the fields of mathematics most generally useful in theoretical computer science. This includes computability theory, computational complexity theory, and information theory. Computability theory examines the limitations of various theoretical models of the computer, including the most powerful known model - the Turing machine. Complexity theory is the study of tractability by computer; some problems, although theoretically solvable by computer, are so expensive in terms of time or space that solving them is likely to remain practically unfeasible, even with rapid advance of computer hardware. Finally, information theory is concerned with the amount of data that can be stored on a given medium, and hence concepts such as compression and entropy.

As a relatively new field, discrete mathematics has a number of fundamental open problems. The most famous of these is the "P=NP?" problem, one of the Millennium Prize Problems.

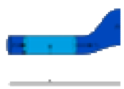
$$\begin{matrix} (1,2,3) & (1,3,2) \\ (2,1,3) & (2,3,1) \\ (3,1,2) & (3,2,1) \end{matrix}$$

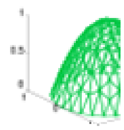
Combinatorics	Theory of computation	Cryptography	Graph theory

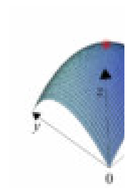
Applied mathematics

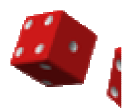
Applied mathematics considers the use of abstract mathematical tools in solving concrete problems in the sciences, business, and other areas. An important field in applied mathematics is statistics, which uses probability theory as a tool and allows the description, analysis, and prediction of phenomena where chance plays a role. Most experiments, surveys and observational studies require the informed use of statistics. (Many statisticians, however, do not consider themselves to be mathematicians, but rather part of an allied group.) Numerical analysis investigates computational methods for efficiently solving a broad range of mathematical problems that are typically too large for human numerical capacity; it includes the study of rounding errors or other sources of error in computation.

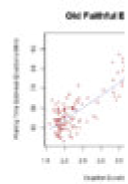

Mathematical physics


Mathematical fluid dynamics

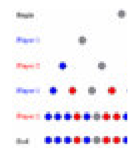

Numerical analysis


Optimization


Probability


Statistics


Financial mathematics


Game theory

Common misconceptions

Mathematics is not a closed intellectual system, in which everything has already been worked out. There is no shortage of open problems. Mathematicians publish many thousands of papers embodying new discoveries in mathematics every month.

Mathematics is not numerology, nor is it accountancy; nor is it restricted to arithmetic.

Pseudomathematics is a form of mathematics-like activity undertaken outside academia, and occasionally by mathematicians themselves. It often consists of determined attacks on famous questions, consisting of proof-attempts made in an isolated way (that is, long papers not supported by previously published theory). The relationship to generally-accepted mathematics is similar to that between pseudoscience and real science. The misconceptions involved are normally based on:

- misunderstanding of the implications of mathematical rigor;
- attempts to circumvent the usual criteria for publication of mathematical papers in a learned journal after peer review, often in the belief that the journal is biased against the author;
- lack of familiarity with, and therefore underestimation of, the existing literature.

The case of Kurt Heegner's work shows that the mathematical establishment is neither infallible, nor unwilling to admit error in assessing 'amateur' work. And like astronomy, mathematics owes much to amateur contributors such as Fermat and Mersenne.

Mathematics and physical reality

Mathematical concepts and theorems need not correspond to anything in the physical world. Insofar as a correspondence does exist, while mathematicians and physicists may select axioms and postulates that seem reasonable and intuitive, it is not necessary for the basic assumptions within an axiomatic system to be true in an empirical or physical sense. Thus, while most systems of axioms are derived from our perceptions and experiments, they are not dependent on them.

For example, we could say that the physical concept of two apples may be accurately modeled by the natural number 2. On the other hand, we could also say that the natural numbers are *not* an accurate model because there is no standard "unit" apple and no two apples are exactly alike. The modeling idea is further complicated by the possibility of fractional or partial apples. So while it may be instructive to visualize the axiomatic definition of the natural numbers as collections of apples, the definition itself is not dependent upon nor derived from any actual physical entities.

Nevertheless, mathematics remains extremely useful for solving real-world problems. This fact led Eugene Wigner to write an essay, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*.